Elliptic monopoles and (4,0)-supersymmetric sigma models with torsion

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ABSTRACT

We explicitly construct the metric and torsion couplings of two-dimensional (4,0)-supersymmetric sigma models with target space a four-manifold that are invariant under a $U(1)$ symmetry generated by a tri-holomorphic Killing vector field that leaves in addition the torsion invariant. We show that the metric couplings arise from magnetic monopoles on the three-sphere which is the space of orbits of the group action generated by the tri-holomorphic Killing vector field on the sigma model target manifold. We also examine the global structure of a subclass of these metrics that are in addition $SO(3)$-invariant and find that the only non-singular one, for models with non-zero torsion, is that of $SU(2) \times U(1)$ WZW model.

1. Introduction

It has been known for sometime that there is an interplay between the number of supersymmetries which leave the action of a sigma model invariant and the geometry of its target space. More recently, sigma models with symmetries generated by Killing vector fields are a fertile area for investigation of the properties of T-duality. The couplings of two-dimensional sigma models are the metric g and a locally defined two-form b on the sigma model manifold $\mathcal{M}$. The closed three-form $H = \frac{3}{2}db$ is called Wess-Zumino term or torsion*. A special class of supersymmetric sigma models are those with (4,0) supersymmetry [1, 2]. These models are ultra-violet finite [1, 3] and arise naturally in heterotic string compactifications [4]. It is known that the target space of (4,0)-supersymmetric sigma models admits three complex structures that obey the algebra of imaginary unit quaternions. When the torsion H vanishes the target space is hyper-Kähler with respect to the metric g. In the presence of torsion, the geometry of the target space of (4,0)-supersymmetric sigma models may not be hyper-Kähler and new geometry arises. It can be shown that given a (4,0)-supersymmetric sigma model with target space a four-dimensional hyper-Kähler manifold with metric g ($H = 0$), it is possible to construct another one with metric $g^F = e^F g$ and torsion $H = {}^*dF$ provided that F is a harmonic *function* with respect to the metric g [4]. The converse though it is not necessarily true. For example the bosonic Wess-Zumino-Witten (WZW) model with target space $SU(2) \times U(1)$ admits a (4,0)-supersymmetric extension [5] but its target manifold is not hyper-Kähler as it can be easily seen by observing that the second deRham cohomology group of $SU(2) \times U(1)$ vanishes. To construct (4,0)-supersymmetric sigma models with torsion that are not conformally related to hyper-Kähler ones, the authors of ref. [6] employed harmonic superspace methods and found a generalisation of the Eguchi-Hanson and Taub-NUT geometries that have *non-zero* torsion. Subsequently, it was shown in [7] that the above

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$\star$ In two-dimensions, the sigma model action can be extended to include terms with couplings other than g and b. For example, the fermionic sector in (p,0)-supersymmetric sigma models has as a coupling a Yang-Mills connection on $\mathcal{M}$.

Eguchi-Hanson geometry with torsion is conformally equivalent to the standard Eguchi-Hanson one.

One result of this letter is to give the most general metric and torsion couplings of a (4,0)-supersymmetric sigma model with target space a four-dimensional manifold $\mathcal{M}$ that admits a tri-holomorphic Killing vector field X that in addition leaves the torsion H invariant. Let $\mathcal{N}$ be the space of orbits of the $U(1)$ group action generated by the vector field X on the sigma model target manifold $\mathcal{M}$ (away from the fixed points of X). We shall show that the non-vanishing components of the metric g and torsion H are given as follows:

$$
\begin{aligned}
ds^2 &= V^{-1}(dx^0 + \omega_i dx^i)^2 + V\gamma_{ij}dx^i dx^j \\
H_{ijk} &= \lambda\epsilon_{ijk}V, \qquad\qquad i,j,k = 1,2,3 ,
\end{aligned}
\tag{1.1}
$$

where ω is a one-form, V is a scalar function and γ is a three-metric on $\mathcal{N}$, so all depend on the co-ordinates $\{x^i; i = 1,2,3\}$ of $\mathcal{N}$, and λ is a real constant. The tensors γ, V and ω satisfy the following conditions:

$$
\begin{aligned}
2\partial_{[i}\omega_{j]} &= \epsilon_{ij}{}^k\partial_k V \\
{}^{(3)}R_{ijkl} &= 2\lambda^2\gamma_{k[i}\gamma_{j]l} ,
\end{aligned}
\tag{1.2}
$$

where ϵ_{ijk} is adopted to the metric γ. The scalar function V is therefore a harmonic function of $\mathcal{N}$ with respect to the metric γ and the metric γ has *positive* constant curvature. The latter implies that the three-manifold $\mathcal{N}$ is elliptic and therefore locally isometric to the (round) three-sphere. In the case that the constant λ is zero one recovers the Gibbons-Hawking metrics [8, 9] that are associated with monopoles on the flat $({}^{(3)}R = 0)$ three-space $\mathcal{N} = \mathcal{R}^3$. In the case that $\lambda \neq 0$ and V constant, we recover the (4,0)-supersymmetric WZW model with target space the manifold $SU(2) \times U(1)$. We shall also study the global properties of a subclass of these metrics that are invariant under an isometry group with Lie algebra $u(2)$ acting on $\mathcal{M}$ with three-dimensional orbits ($u(2)$-invariant metrics), for $\lambda \neq 0$, and we shall find that apart from the metric of $SU(2) \times U(1)$ WZW model all the remaining ones are *singular*.

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The material of this letter is organised as follows: In section two, the geometry of (4,0)-supersymmetric sigma models is briefly reviewed. In section three, the derivation of equations (1.1) and (1.2) is presented. In section four, the singularity structure of a subclass of the metrics (1.1) that are $u(2)$-invariant is investigated. Finally in section five we give our conclusions.

2. Geometry and (4,0) supersymmetry

Let $\mathcal{M}$ be a Riemannian manifold with metric g and a locally defined two-form b. The action of the (1,0)-supersymmetric sigma model with target space $\mathcal{M}$ is

$$I = -i \int d^2x d\theta^+ (g_{\mu\nu} + b_{\mu\nu}) D_+\phi^\mu \partial_= \phi^\nu \, , \tag{2.1}$$

where $(x^{\mathbin{\text{\ss}}}, x^=, \theta^+)$ are the co-ordinates of (1,0) superspace $\Xi^{(1,0)}$ and D_+ is the supersymmetry derivative $(D_+^2 = i\partial_{\mathbin{\text{\ss}}})$; $(x^{\mathbin{\text{\ss}}}, x^=) = (x+t, t-x)$ are light-cone co-ordinates. The fields ϕ of the sigma model are maps from the (1,0) superspace, $\Xi^{(1,0)}$, into the target manifold $\mathcal{M}$.

To construct sigma models with (4,0) supersymmetry, we introduce the transformations

$$\delta_I \phi^\mu = a_{-r} I_r{}^\mu{}_\nu D_+\phi^\nu \tag{2.2}$$

written in terms of (1,0) superfields, where I_r, r=1,2,3, are (1,1) tensors on $\mathcal{M}$ and a_{-r} are the parameters of the transformations. The commutator of these transformations closes to translations* provided that

$$I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t$$
$$N(I_r)^\mu{}_{\nu\rho} = 0 \, , \tag{2.3}$$

$\star$ Due to the classical superconformal invariance of the model the parameters a_{-r} of the super-symmetry transformations can be chosen to be semi-local in which case the commutator of supersymmetry transformations closes to translations and supersymmetry transformations.

where

$$N(I_r)^\mu{}_{\nu\rho} = I_r{}^\kappa{}_\nu \partial_\kappa I_r{}^\mu{}_\rho - I_r{}^\mu{}_\kappa \partial_\nu I_r{}^\kappa{}_\rho - (\rho \leftrightarrow \nu) \tag{2.4}$$

is the Nijenhuis tensor of I_r. The conditions (2.3) imply that I_r, $r = 1, 2, 3$, are complex structures that satisfy the algebra of imaginary unit quaternions.

The action (2.1) of (1,0)-supersymmetric sigma model is invariant under the (4,0) supersymmetry transformations (2.2) provided that, in addition to the conditions obtained above for the closure of the algebra of these transformations, the following conditions are satisfied:

$$g_{\kappa(\mu} I_r{}^\kappa{}_{\nu)} = 0 , \qquad \nabla^{(+)}_\mu I_r{}^\nu{}_\rho = 0 , \tag{2.5}$$

where $\nabla^{(\pm)}$ is the covariant derivative of the connection

$$\Gamma^{(\pm)\mu}{}_{\nu\rho} = \{^\mu_{\nu\rho}\} \pm H^\mu{}_{\nu\rho} , \tag{2.6}$$

and

$$H_{\mu\nu\rho} = \frac{3}{2} \partial_{[\mu} b_{\nu\rho]} . \tag{2.7}$$

Note that if I_r are integrable, $N(I_r) = 0$, and covariantly constant with respect to the $\nabla^{(+)}$ covariant derivative, $\nabla^{(+)} I_r = 0$, then the torsion H is (2,1)-and (1,2)-form with respect to all complex structures I_r. Note also that, if the torsion H is zero, the above conditions simply imply that the sigma model target manifold is hyper-Kähler with respect to the metric g

Next consider the transformations

$$\delta\phi^\mu = \epsilon^a X_a^\mu(\phi) , \tag{2.8}$$

of the sigma model field ϕ, where $\{\epsilon^a; a = 1, 2, \ldots\}$ are the parameters of these transformations and $\{X_a; ; a = 1, 2, \ldots\}$ are vector fields on the sigma model manifold $\mathcal{M}$. The action (2.1) is invariant under these transformations up to surface

terms provided that

$$\nabla_\mu X_{a\nu} + \nabla_\nu X_{a\mu} = 0, \qquad X_a^\kappa H_{\kappa\mu\nu} = \partial_{[\mu} u_{a\nu]} \ . \tag{2.9}$$

These conditions imply that $\{X_a; ; a = 1, 2, \ldots\}$ are Killing vector fields on the sigma model manifold $\mathcal{M}$ and leave the closed three-form H invariant . The commutator of the (2.8) transformations with the (4,0) supersymmetry transformations is

$$[\delta_\epsilon, \delta_I]\phi^\mu = \epsilon^b a_{-r} \mathcal{L}_{X_b} I_r{}^\mu{}_\nu D_+ \phi^\nu \tag{2.10}$$

This commutator closes on the existing symmetries of the theory, if we take

$$\mathcal{L}_{X_a} I_r{}^\mu{}_\nu = 0 \ , \tag{2.11}$$

where $\mathcal{L}_{X_a}$ is the Lie derivative with respect to the vector field X_a. All three complex structures I_r are invariant under the vector field X_a, *i.e.* X_a is tri-holomorphic. Note that it is possible to relax the above condition. For example, we can take that the complex structures I_r rotate under the isometries but this possibility will not be considered here. Finally the commutator of the transformations (2.8) with themselves closes provided that

$$[X_a, X_b] = f_{ab}{}^c X_c \tag{2.12}$$

where $f_{ab}{}^c$ are the structure constants of a Lie algebra.

3. (4,0) supersymmetry and four-dimensional geometry

To find the geometry of the target space of sigma models with (4,0) super-symmetry, one has to solve the conditions (2.3) and (2.5) of the previous section. For this we will restrict ourselves to four-dimensional target spaces $\mathcal{M}$ and we will assume that there is a Killing vector field X on $\mathcal{M}$ that leaves the torsion H and the complex structures I_r invariant. As we have seen in the previous section, these conditions on the vector field X are those required for the invariance of the action (2.1) under the transformations (2.8) of the sigma model fields ϕ and the closure of the commutator these transformations with the (4,0) supersymmetry transformations.

Next we adopt coordinates $\{x^0, x^i; i = 1, 2, 3\}$ on the sigma model manifold along the Killing vector field X, *i.e.*

$$X = \frac{\partial}{\partial x^0} \ . \tag{3.1}$$

Then the metric g and the torsion H are written as follows:

$$ds^2 = V^{-1}(dx^0 + \omega_i dx^i)^2 + V \gamma_{ij} dx^i dx^j \ , \tag{3.2}$$

and

$$H_{0ij} = \partial_{[i} u_{j]}, \qquad H_{ijk} = \epsilon_{ijk} U \ , \tag{3.3}$$

where γ, u, V and U are tensors of the space of orbits, $\mathcal{N}$, of the group action generated by the Killing vector field X on $\mathcal{M}$ (away from the fixed points of X) and depend only upon the co-ordinates $\{x^i; i = 1, 2, 3\}$. The tensor γ is a metric on $\mathcal{N}$ and the tensor ϵ_{ijk} is adopted to the three-metric γ.

We introduce the frame

$$
\begin{aligned}
e^{\underline{0}} &= V^{-\frac{1}{2}}(dx^0 + \omega_i dx^i) \ , \\
e^{\underline{r}} &= V^{\frac{1}{2}}E_i^r dx^i \ , \qquad \underline{r} = 1, 2, 3
\end{aligned}
\tag{3.4}
$$

associated with the metric (3.2), and its dual

$$
\begin{aligned}
e_{\underline{0}} &= V^{\frac{1}{2}}\partial_0 \\
e_{\underline{r}} &= V^{-\frac{1}{2}}E_r^i\left(\partial_i - \omega_i \partial_0\right) \ ,
\end{aligned}
\tag{3.5}
$$

where $\partial_0 = \frac{\partial}{\partial x^0}$, $\partial_i = \frac{\partial}{\partial x^i}$, and $E^{\underline{r}}$ and $E_{\underline{r}}$ is a frame of the metric γ and its dual, respectively. Next we introduce three invariant (1,1) tensors on $\mathcal{M}$ as follows:

$$
I_r = e_{\underline{0}} \otimes e^{\underline{r}} - e_{\underline{r}} \otimes e^{\underline{0}} - \epsilon_{rst}e_{\underline{s}} \otimes e^{\underline{t}}
\tag{3.6}
$$

Contracting with the metric g, we can show that

$$
\mathcal{I}_r \equiv \gamma_{\mu\kappa}I_r{}^{\kappa}{}_{\nu}dx^{\mu} \otimes dx^{\nu} = 2e^{\underline{0}} \wedge e^{\underline{r}} - \epsilon_{rst}e^{\underline{s}}e^{\underline{t}} \ .
\tag{3.7}
$$

are two-forms on $\mathcal{M}$ and so the first condition of eqn. (2.5) is satisfied. It is straightforward to verify that the tensors I_r are almost complex structures that satisfy the algebra of imaginary unit quaternions and so the first condition of eqn. (2.3) is also satisfied. Furthermore, it can be shown that $\{I_r; r = 1, 2, 3\}$ (eqn. (3.6)) is the most general set of almost complex structures, up to an $SO(3)$ (gauge) rotation of the frame $E^{\underline{r}}$ of the three-metric γ, that obey the algebra of imaginary unit quaternions and are invariant under the group action generated by the Killing vector field X. Therefore X is a tri-holomorphic vector field and satisfies eqn. (2.11). So it remains to find the conditions on V, U, u and γ in order the metric (3.2), antisymmetric tensor (3.3) and the almost complex structures I_r satisfy the second condition of (2.3) and the second condition of (2.5). Combining

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the second condition of (2.5) and (2.11), we can show that

$$\nabla_i^{(+)} X_j \tag{3.8}$$

is a (1,1)-form with respect to all almost complex structures I_r which in turn implies that

$$2\partial_{[i}\omega_{j]} - 2V\partial_{[i}u_{j]} = \epsilon_{ij}{}^k\partial_k V . \tag{3.9}$$

Next using the fact that the torsion H is (2,1)-and (1,2)-form with respect to all almost complex structures I_r, the condition

$$\nabla_{[\mu}^{(+)} \mathcal{I}_{\nu\rho]}^r = 0 \tag{3.10}$$

implies that

$$du = 0, \qquad UV^{-1} = \lambda,$$
$$^{(3)}R_{ijkl} = 2\lambda^2 \gamma_{k[i}\gamma_{j]l} , \tag{3.11}$$

where $^{(3)}R$ is the curvature of the metric γ and λ is a real constant. After some computation, it can be shown that the almost complex structures are integrable without further conditions, *i.e.* their Nijenhuis tensor vanishes. Finally, the second condition of (2.5) follows from the integrability of I_r, the condition (3.10) and the fact that H is (2,1)- and (1,2)-form with respect to all complex structures I_r. Substituting the equations (3.9) and (3.11) back into (3.2) and (3.3) we get the metric g and torsion H of eqn. (1.1).

Apart from the special cases for which either $\lambda = 0$ or V constant mentioned in the introduction, new metrics can be found by taking

$$V(x^i) = c_0 + \sum_{n=1}^N c_n G(x, x_n) \tag{3.12}$$

where $\{G(x, x_n); n = 1, \ldots, N\}$ are the Green's functions of the Laplace-Beltrami operator Δ associated with the metric γ, $\{c_0, c_n; n = 1, \ldots, N\}$ are real constants and $\{x_n; n = 1, \ldots, N\}$ are N points in $\mathcal{N}$.

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The metric g of (1.1) is not conformally equivalent to a hyper-Kähler one if $\lambda \neq 0$ and x^0 is an angular co-ordinate. To show this, let

$$g^F = e^F g \tag{3.13}$$

where F is a function of $\mathcal{M}$. Since I_r are complex structures, to prove that g^F is hyper-Kähler it is enough to show that the three two-forms $\mathcal{I}^F_{r\ \mu\nu} \equiv g^F_{\mu\kappa} I_r{}^\kappa{}_\nu$ are closed, i.e. $d\mathcal{I}^F_r = 0$. After some computation, it can be shown that $d\mathcal{I}^F_r = 0$ implies that

$$\partial_0 F = -2\lambda V^{-1}$$
$$\partial_i F = -2\lambda V^{-1}\omega_i \ . \tag{3.14}$$

It is clear that, if $\lambda = 0$ (the torsion H is zero), the most general solution of (3.14) is that F is equal to a real constant. In this case g is hyper-Kähler and so is g^F. In the case that $\lambda \neq 0$, we differentiate the second equation in (3.14) with respect to ∂_0 and we get

$$\partial_0 \partial_i F = 0 \tag{3.15}$$

but

$$\partial_0 \partial_i F \equiv \partial_i \partial_0 F = -2\lambda \partial_i V^{-1} = 0 \ , \tag{3.16}$$

which implies that V is constant. So for the metric g to be conformally equivalent to a hyper-Kähler metric, V must be a real constant. However, the case that V equal to a constant can also be excluded if the co-ordinate x^0 is an angle. To see this observe that all solutions F of (3.14) are linear in x^0 and so they cannot be scalar functions of $\mathcal{M}$. For example, the metric g of the (4,0)-supersymmetric WZW model with target space the group $SU(2) \times U(1)$ is not conformally equivalent to a hyper-Kähler one. Using a similar argument and under the assumptions that $\lambda \neq 0$ and x^0 is an angle, we can also show that the metric g of eqn (1.1) is not conformally equivalent to a Kähler one.

4. $u(2)$-invariant metrics

The metrics given in eqn. (1.1) may have singularities. To examine such global properties of these metrics, we consider the following example. Let us write the three-metric γ in spherical polar co-ordinates

$$d\Omega_3^2 = R^2 d\psi^2 + \sin^2 \psi d\Omega_2^2 \qquad (4.1)$$

where R is the radius of the three-sphere and

$$d\Omega_2^2 = R^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right) \qquad (4.2)$$

is the metric on a two-sphere. Next consider the case that V is equal to the Green's function G that depends only on ψ. Such V is

$$V = c_1 \cot \psi + c_0 \qquad (4.3)$$

where c_1 and c_0 are real constants. Then it can be arranged that

$$\omega = c_1 \cos \theta d\varphi \ . \qquad (4.4)$$

The metric g with V given in (4.3) admits an isometry group with Lie algebra $u(2)$ acting on $\mathcal{M}$ with three-dimensional orbits. We will consider three cases the following. Case(i) the constant $c_0 = 0$ and the constant $c_1 \neq 0$, in this case the metric g exhibits singular behaviour at $\psi = 0$ and $\psi = \frac{\pi}{2}$. The singularity at $\psi = 0$ is a nut singularity and can be removed by the standard methods of ref. [10] provided that $0 \leq \frac{x^0}{c_1 R} < 4\pi$ ($c_1 > 0$). However the singularity at $\psi = \frac{\pi}{2}$ cannot be removed as it can be easily seen by changing the ψ co-ordinate to $u = \cot \psi$ and by studying the behaviour of the metric at $u = 0$. Case (ii) the constants $c_0 \neq 0$ and $c_1 \neq 0$, in this case it can be arranged by rescaling the metric to set $c_0 = 1$. The metric g has singular behaviour at $\psi = 0$ and $\cot \psi = -\frac{1}{c_1}$. The singularity at

$\psi = 0$ is a nut singularity and again it can be removed provided that $0 \le \frac{x^0}{c_1 R} < 4\pi$ ($c_1 > 0$) as in case (i). But the singularity at $\cot \psi = -\frac{1}{c_1}$ cannot be removed as it can be easily seen by changing the ψ co-ordinate to $u = \cot \psi + \frac{1}{c_1}$ and by studying the behaviour of the metric at $u = 0$. Another way to see that the metric is singular at $\cot \psi = -\frac{1}{c_1}$ is to observe that the geodesics of $\mathcal{M}$ that depend only upon ψ reach the singularity at finite proper time and they cannot be extended beyond it. The manifold $\mathcal{M}$ is therefore geodesically incomplete.

Next define the new co-ordinate $s = \tan \psi$, then, the metric g is rewritten as

$$ds^2 = (c_0 + c_1 s^{-1})^{-1} \left(dx^0 + c_1 \cos \theta d\varphi \right)^2 + (c_0 + c_1 s^{-1}) \left[\frac{R^2}{1+s^2} ds^2 + \frac{s^2}{1+s^2} d\Omega_2^2 \right] . \tag{4.5}$$

This metric as $s \to +\infty$ (or $\psi \to \frac{\pi}{2}$) behaves as

$$ds^2 \sim R^2 dw^2 + c_1^2 R^2 \left[(d(\frac{x^0}{Rc_1}) + \cos \theta d\phi)^2 + \frac{1}{c_1^2} (d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{4.6}$$

where $w = s^{-1}$. Note that, if $c_1 = 1$, the manifold $\mathcal{M}$ near $\psi = \frac{\pi}{2}$ becomes $\mathcal{R} \times S^3$ and S^3 has radius $2R$ but eventually the metric g will become singular at $\cot \psi + 1 = 0$ as we have mentioned above. The metric g in the parameterisation (4.5) involving the co-ordinates $\{s, x^0, \theta, \psi\}$ is the same as the metric given by the authors of ref. [6] (up to a gauge choice for the one-form ω, and a relabelling of some of parameters of the metric and some of the co-ordinates of $\mathcal{M}$) for the Taub-NUT geometry with torsion. The same authors had also observed that the space of orbits $\mathcal{N}$ for this metric is the three-sphere. But our conclusion that this metric is singular is opposite from that of ref. [6]. The torsion is

$$H = -\lambda c_1 R^3 \sin^2 \psi \sin \theta (c_1 \cot \psi + c_0) d\psi d\theta d\phi , \tag{4.7}$$

and it is non-singular. Finally, case (iii) the constant $c_0 \ne 0$ and the constant $c_1 = 0$, in this case the metric g and the torsion H become those of the $SU(2) \times U(1)$ WZW model.

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5. Concluding Remarks

We have determined the metric and torsion couplings of (4,0)-supersymmetric sigma models with a Noether symmetry generated by a tri-holomorphic Killing vector field and target space a four-dimensional manifold. These metrics are naturally associated to monopoles on the three-sphere which is the space of orbits of the group action generated by the tri-holomorphic Killing vector field on the sigma model target manifold. In relation to this result, it is worth pointing out that the hyper-Kähler Gibbons-Hawking metrics are associated with monopoles on the flat three-space, and the scalar flat Kähler LeBrun metrics [11] are associated with monopoles on the hyperbolic three-space. We have also studied the global properties of a subclass of these metrics that admit an isometry group with Lie algebra $u(2)$ acting on the sigma model manifold with three-dimensional orbits, for models with non-zero torsion, and we have found that, apart from the metric of the WZW model with target space the group $SU(2) \times U(1)$, are singular. It also seems likely that all the remaining metrics, for models with non-zero torsion, are singular as well.

The class of the above metrics that are singular cannot be thought as gravitational instantons associated with some string inspired extension of the Einstein gravity. However they may serve as couplings of supersymmetric sigma models way from the singularities. Such sigma models will be ultra-violet finite by the arguments of refs. [1, 3]. It may also be that the (4,0) supersymmetry of the above sigma models can be extended to a (4,4) one by an appropriate addition of a fermionic sector. This suggestion is supported by the fact that the (4,0)-supersymmetric WZW model with target space the group $SU(2) \times U(1)$ admits such an extension.

The T-duality transformation can be easily applied to the couplings g and b of the (4,0)-supersymmetric sigma model with target space a four-dimensional manifold with respect to the tri-holomorphic Killing vector field X to give the couplings g', b' and dilaton of the dual model [12] (see also [13]). In fact the couplings g' and

b' can be simplified in this case because, as we have shown in section 3, $X \cdot H = 0$. In addition, the T-dual theories of the above (4,0)-supersymmetric sigma models are expected to be ultra-violet finite since the latter are ultra-violet finite. Finally, the application of the new metrics that we have derived from consideration of (4,0)-supersymmetric models to string theory and conformal field theory needs further study. Such an investigation will involve the construction of the associated superconformal field theory.

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